

# On Time and Space in Higher Order Boolean Circuits

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It is well known that the usual complexity measures of time and space (defined using Turing machines) are strongly related to *size* and *depth* of Boolean circuits. The size of a circuit is the number of its gates; the depth is the longest path in the circuit, seen as a directed acyclic graph (from inputs to outputs). Given a family  $(C_n)_{n \in \mathbb{N}}$  of Boolean circuits (on the standard fan-in 2 basis  $\{\neg, \wedge, \vee\}$ ), such that  $C_n$  has  $n$  inputs and 1 output, we say that it *decides* a language  $L \subseteq \{0, 1\}^*$  if, for all  $x \in \{0, 1\}^n$ ,  $x \in L$  iff  $C_{|x|}(x) = 1$ . We then denote by  $\text{SIZE}(f)$  (resp.  $\text{DEPTH}(f)$ ) the class of languages decided by families of circuits  $(C_n)_{n \in \mathbb{N}}$  such that the size (resp. depth) of  $C_n$  is bounded by  $f(n)$ .

With the above definitions, we have (see [14, 4]):

**Theorem 1 (Fisher and Pippenger, Borodin)**

1.  $\text{TIME}(f) \subseteq \text{SIZE}(O(f \log f))$ ;
2.  $\text{SIZE}(f) \subseteq \text{TIME}(O(f))$ ;
3.  $\text{NSPACE}(f) \subseteq \text{DEPTH}(O(f^2))$ ;
4.  $\text{DEPTH}(f) \subseteq \text{SPACE}(O(f))$ .

In (2) and (4), some notion of *uniformity* on circuit families must be assumed of course. The details are irrelevant for our purposes; it suffices to say that a family is uniform if its circuits may be generated by a program of small complexity (say, logspace).

Our purpose is to investigate what happens in the higher order world, which is the preferred setting for the theory of programming languages. In particular:

- (a)  $\text{TIME}(\cdot)$  and  $\text{SPACE}(\cdot)$  must be replaced by notions of time and space complexity for  $\lambda$ -terms;
- (b) a suitable notion of higher-order Boolean circuit must be introduced, and the corresponding notions of size and depth defined for it.

For what concerns time, one may resort to the number of head-reduction steps, which was proved by Accattoli and Dal Lago [1] to be an invariant cost measure, *i.e.*, if we denote by  $\lambda\text{TIME}(f)$  the class of languages decided (using *e.g.* Church binary strings and Church Booleans) by  $\lambda$ -terms in at most  $f(n)$  steps of head reduction for inputs of size  $n$ , then:

**Theorem 2 (Accattoli and Dal Lago)**

1.  $\text{TIME}(f) \subseteq \lambda\text{TIME}(O(f))$ ;
2.  $\lambda\text{TIME}(f) \subseteq \text{TIME}(O(f^k))$  for some constant  $k$ .

The above tells us we have a good notion of time complexity for  $\lambda$ -terms. When it comes to space complexity, however, much less is known. The ideal would be to find an abstract measure, something that is as much as possible machine-independent. Resorting to notions which are purely internal to the theory of the  $\lambda$ -calculus should guarantee a good level of abstraction (counting head-reductions is a perfect example). The work of Belloch et al. [16] is a very interesting proposal, although it still relies on a low-level description of  $\lambda$ -terms, which is not entirely satisfactory from our point of view.

Let us temporarily ignore the issue and let us address directly point (b): what is a higher-order Boolean circuit? We believe that this should be taken to coincide with *linear  $\lambda$ -terms*. There are several reasons for this choice: like Boolean circuits, linear  $\lambda$ -terms may only compute finite functions; like Boolean circuits, the (sequential) runtime of a linear  $\lambda$ -term coincides with its size; finally, Boolean circuits may be seen as morphisms of a free symmetric monoidal category (in fact, a PROP), while (normal) linear  $\lambda$ -terms are morphisms in a free *closed* symmetric monoidal category. In fact, for technical rea-

sons it is better to consider *affine*  $\lambda$ -terms (erasing is permitted, not duplication), which are a minor variant still adhering to the above picture.

The relationship between affine  $\lambda$ -terms and general  $\lambda$ -terms may be refined by formalizing the intuition (already present in [7]) that the affine  $\lambda$ -calculus is “dense” in the full  $\lambda$ -calculus. This is the object of [8], which may be informally summarized as follows:

- there is a notion of *affine approximation*  $t \sqsubset M$  of a  $\lambda$ -term by affine terms  $t$ , such that  $M$  may be seen as the limit of its affine approximations;
- reduction is continuous: if  $M \rightarrow^* N$ , then for all  $u \sqsubset N$  there exists  $t \sqsubset M$  such that  $t \rightarrow^* u$ .

Although this looks like a good start, we immediately run into trouble: linear-step computations in the  $\lambda$ -calculus may require exponentially large affine approximations. In other words,  $\lambda\text{TIME}(f)$  is *not* included in  $\lambda\text{SIZE}(O(f^k))$  for any constant  $k$ , where we consider the size of affine terms to be the usual one. A possible solution, which we started to explore, is to shift to the *parsimonious*  $\lambda$ -calculus, introduced by the author [9, 10, 13]. Another solution is to consider *linear explicit substitutions*, as in [1, 2]. In both cases, we obtain a result corresponding to points 1 and 2 of Theorem 1. For instance, if  $\text{p}\lambda\text{TIME}(f)$  is the analogue of  $\lambda\text{TIME}(f)$  for the parsimonious  $\lambda$ -calculus, we have

**Theorem 3 ([12])**

1.  $\text{p}\lambda\text{TIME}(f) \subseteq \lambda\text{SIZE}(O(f^k))$  for a constant  $k$ ;
2.  $\lambda\text{SIZE}(f) \subseteq \text{p}\lambda\text{TIME}(O(f^2))$ .

In ongoing work, we are seeking to complete the picture by finding the analogue of points 3 and 4 of Theorem 1, taking the parsimonious  $\lambda$ -calculus as the underlying language. Our approach is based on an interesting correspondence between intersection types and affine approximations, described recently by the author [11]: a  $\lambda$ -term  $M$  has an intersection type iff there exists a simply-typable  $t$  such that  $t \sqsubset M$ . More precisely, one may build a non-idempotent, non-commutative intersection type system (equivalent to the standard one for what concerns typability) such that its types are isomorphic to the simple types for affine terms and its derivations are isomorphic to simply-typed affine terms and, furthermore, an affine term corresponding to a type derivation for

$M$  is actually an approximation of  $M$ . So, we may write  $\Gamma \vdash t \sqsubset M : A$  to say that  $M$  is typable (in intersection types) of type  $A$  with a derivation isomorphic to  $t$  (whose simple type is also  $A$ ). Thanks to this correspondence, which holds also for the parsimonious  $\lambda$ -calculus, and the idea, originally due to Schöpp [15], of using the geometry of interaction to perform space-efficient computation on  $\lambda$ -terms, we obtain

**Theorem 4** *Let  $M$  be a parsimonious  $\lambda$ -term s.t. there exists a directed (w.r.t. the approximation order) sequence  $(t_n)_{n \in \mathbb{N}}$  s.t., for all  $n \in \mathbb{N}$ ,*

$$\vdash t_n \sqsubset M : \text{Str}_n \multimap \text{Bool},$$

*where  $\text{Str}_n$  are (bigger and bigger) instances of the usual type of binary strings, and let  $d_n$  be the depth of  $\text{Str}_n$  (as a syntactic tree). Then,  $M$  decides a language in  $\text{SPACE}(O(d_n \log |t_n|))$ .*

The above result suggests that, if we take higher-order Boolean circuits to be affine terms of type  $\text{Str}[] \multimap \text{Bool}$  (with  $\text{Str}[]$  some instance of the type of binary strings), then we should take as “depth” the depth of the formula  $\text{Str}[]$ . This is in accord with what Terui did for Boolean proof nets [17] and agrees, up to a multiplicative instead of additive factor, with what Borodin [4] found for Turing machines and (first-order) Boolean circuits: a uniform family of circuits of depth  $d_n$  and size  $s_n$  decides a language in  $\text{SPACE}(d_n + \log s_n)$  (this is how point 4 of Theorem 1 is proved).

Observe that Theorem 4 may allow to bypass the definition of a space measure for parsimonious  $\lambda$ -terms: the existence of a family of suitable typings would constitute *per se* an abstract, machine-independent measure. The relationship between intersection types and time complexity of  $\lambda$ -terms was well known [6, 3, 18, 5]; it is interesting to see that it may work for space, too. In the talk, we will discuss and explain more in detail the above results and the perspectives they offer.

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